Presentations for certain finite quaternionic reflection groups

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1. Introduction

One of Coxeter's highly remarkable discoveries is that the diagrams bearing his name can be interpreted as presentations (by means of generators and relations) for real linear groups generated by reflections having roots (that is, -1 eigenvalues) at angles indicated by the diagram.

Given such a finite linear reflection group, there is a natural and canonical way to isolate a fundamental domain in the reflection space bounded by reflecting hyperplanes. The associated diagram is then obtained by taking for nodes these hyperplanes, joining two nodes if the reflections corresponding to these hyperplanes do not commute and labelling the resulting edge by the order of the product of these two reflections. (This number is of course directly related to the angle between the roots and also the angle between the two hyperplanes.) Conversely, given a diagram, a realization by means of hyperplanes having the right angles leads to a reflection group, namely the group generated by all reflections in these hyperplanes.

In 1953, Shephard (1953) provided similar diagrams for finite unitary reflection groups. The fundamental domain no longer played a role, but the nodes still represented roots of a generating set of reflections, edges still corresponded to non-orthogonal roots (and therefore non-commuting reflections), and the edge labelling kept its meaning. The Coxeter diagrams of finite real reflection groups are free of circuits. Shephard's connected diagrams corresponding to unitary reflection groups contain a single circuit, which is a triangle.

Fourteen years later, Coxeter (1967) provided presentations for these unitary reflection groups using Shephard's diagrams. The presentation for the Coxeter group related to these diagrams had to be extended by a single relation coming from the triangle (here we assume, without loss of generality, that the diagram is connected - otherwise study of the group and its reflection representation can be reduced to the factors of a direct product decomposition). Upon adding a number to the diagram that is to be interpreted as a label of the triangle, the presentation of the corresponding reflection group could still be read off from the diagram.

Now that Hoggar (1982) has provided diagrams for the most interesting finite quaternionic reflection groups, the question arises of producing a presentation that can, again, be read off directly from the diagram. In the present note, we report on some attempts to this end.

For each of the groups under study, we exhibit a presentation, largely based on some rules of thumb for reading off relations from the diagram. One of these is a relation obtained from identifying subgroup centres which McMullen once suggested to Coxeter for unitary groups; I am grateful to Leonard Soicher for pointing out such a rule to me for the quaternionic case. Two presentations remain quite peirastic: we have found no proper mnemonic device to distill a presentation from the diagram.

The presentations are useful in the study of subgroups generated by reflections corresponding to subsets of nodes of the diagram: presentations for these subgroups can be obtained by removal of all relations involving a reflection corresponding to an excluded node. Thus, we provide a good starting point for the geometric study of the permutation representations on these and (other) reflection subgroups. Extrapolating from the complex case, one may also venture to predict a use of these presentations for a possible classification of discrete quaternionic groups.

Two out of the seven groups (viz. $W(S_1)$ and $W(U)$) are 3-transposition groups: the order of the product of any two reflections is either 2 or 3. In this context, the presentations for these two groups described below are well known, see for instance Hall (1990) and Zara (1985).

2. Preliminaries

A *quaternionic reflection group* is a quaternionic linear group, that is, a subgroup of the general quaternionic linear group $GL(n, \mathbf{D})$ for some integer n generated by *reflections* (that is, elements of $GL(n, \mathbf{D})$ having a fixed space of dimension n -1); the number n will be referred to as the *dimension* of the reflection group. We recall that a subgroup G of $GL(n, \mathbf{D})$ is called *primitive* if the only set $\{V_1, ..., V_t\}$ of subspaces $\mathbf{D}^n$ decomposing $\mathbf{D}^n$ and stable under G occurs for $t = 1$.

The group G is called *complex* or *real* if it is conjugate in $GL(n, \mathbf{D})$ to a subgroup of $GL(n, \mathbf{C})$ or $GL(n, \mathbf{R})$, respectively. Finite real groups are dealt with extensively in Bourbaki (1968), complex reflection groups have been classified in Shephard and Todd (1954) (see also Cohen (1976)). As finite complex linear groups leave invariant a unitary form, they are also called unitary instead of complex.

The finite quaternionic groups have been classified in Cohen (1980). In studying presentations, we shall content ourselves with considering the most interesting examples, namely those finite quaternionic reflection groups (of dimension $\geq$ 3) whose complexifications are primitive complex linear groups. There are seven of them. Hoggar (1982) has worked out certain minimal sets of generating reflections for these groups and has drawn the diagrams based on the corresponding roots. Here, we shall provide presentations of these groups on the basis of these (labelled) graphs.

3. The groups involved

The primitive quaternionic reflection groups of dimension ≥ 3 whose complexifications are primitive are the following (cf. Cohen 1980):

dim n	root system	reflection group	its order	selected subgroup	its index
3	Q	$2 \times PSU(3,3)$	$2^6 \cdot 3^6 \cdot 7$	$W(J_3(4))$	36
3	R	$2 \cdot HJ$	$2^8 \cdot 3^3 \cdot 5^2 \cdot 7$	$W(J_3(5))$	560
4	S_1	$(D_2 \circ D_8 \circ D_8) \cdot G(3,3,3)$	$2^8 \cdot 3^3$	$G(3,3,3)$	128
4	S_2	$(D_2 \circ D_8 \circ D_8) \cdot G(3,3,4)$	$2^{10} \cdot 3^4$	$G(4,4,3)$	864
4	S_3	$(D_2 \circ D_8 \circ D_8) \cdot \Omega^-(6,2)$	$2^{13} \cdot 3^4 \cdot 5$	$W(S_2)$	40
4	T	$(\circ^3 SL(2,5)) Sym_3$	$2^8 \cdot 3^4 \cdot 5^3$	$G(5,5,3)$	17280
5	U	$2 \times PSU(5,2)$	$2^{11} \cdot 3^5 \cdot 5 \cdot 11$	$W(S_1)$	3960

Here, each line corresponds to a quaternionic reflection group. Its dimension is listed in the first column, the name given in Cohen (1980) for its 'root system', that is, a suitably chosen set of root reflections (at least one for each reflection) in the group, is listed next. Analogously to the real case, the reflection group corresponding to the root system X is denoted by $W(X)$ (see the fifth column for examples). The third column contains a description of the isomorphism types; $\circ^3 SL(2, 5)$ stands for the central product of three copies of the special linear group $SL(2, 5)$ on the 2-dimensional vector space over the field with 5 elements. The fourth column provides a reflection subgroup with respect to which a presentation will be given below.

The diagrams found by Hoggar (1982) are depicted in Figure 1. For the diagrams associated with S_2 and S_3 we have made a slight adaption; Hoggar's diagrams for those two cases can be obtained from ours by replacing the reflection c with aca. The absence of a label at an edge or triangle is to be interpreted as the label 3.

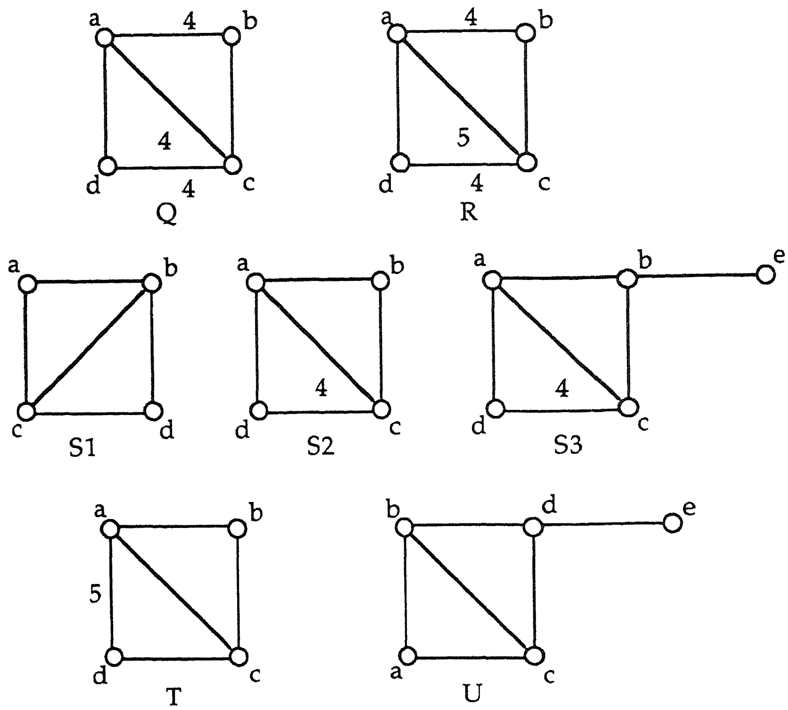

Figure 1.

4. The complex reflection groups

Let us briefly recall Coxeter's 'complex' results (cf. Coxeter (1967)). The usual Coxeter group presentation related to these diagrams had to be extended by a single relation coming from the triangle (here we assume, without loss of much generality, that the diagram is connected - otherwise study of the group and its reflection representation can be reduced to the factors of a direct product decomposition). To this end an additional number was introduced that could be interpreted as the label of the triangle. Now, if the reflections corresponding to the nodes of the diagram are denoted by a, b, c and the triangle label is m, the additional relation reads

$$(abcb)^m = 1.$$

Since at least two edges of the triangle had label 3, the additional relation is independent of the order in which the nodes are read off from the diagram. We provide the two most important series as examples.

4.1 $J_3(m)$

Here, only $m = 4, 5$ are relevant. The above rule leads to the following presentation of $W(J_3(m))$:

generators: $a, b, c;$

relations: $a^2 = b^2 = c^2 = (ab)^4 = (ac)^3 = (bc)^3 = 1,$

$$(abcb)^m = 1.$$

Such a presentation is checked by use of a Todd-Coxeter coset enumeration[1]. Here the enumeration has taken place with respect to the subgroup $\langle a, b\rangle$.

If $m = 4$, coset enumeration outputs 42 cosets. The element $(abc)^7$ has order 2 and lies in the centre.

If $m = 5$, the output yields 270 cosets. The group is then isomorphic to the central extension $6{\cdot}Alt_6$ of the alternating group on 6 letters. Its centre is cyclic of order 6. The element abc has order 30. The element $(abc)^5$ generates the centre.

4.2 $G(m, m, 3)$

The imprimitive complex reflection groups we shall need are $G(m, m, 3)$ for $m = 3, 4, 5$ in the notation of Shephard and Todd (1954). They have the following presentation.

generators: $a, b, c;$

relations: $a^2 = b^2 = c^2 = (ab)^3 = (ac)^3 = (bc)^m = 1,$

$$(abcb)^3 = 1.$$

Thus, the underlying diagram is a triangle one of whose sides has label m, while the label of the triangle equals 3. If $m = 3$, the centre has order 3 and is generated by $(abc)^2$.

4.3 Five generators in four dimensions

Among the primitive complex reflection groups, there is a single 4-dimensional one that cannot be generated by 4 reflections. A diagram corresponding to five generating reflections is given in Figure 2.

[1]The computations have been performed in CAYLEY and MAPLE on the computers of t Dutch Computer Algebra Centre of Expertise CAN.

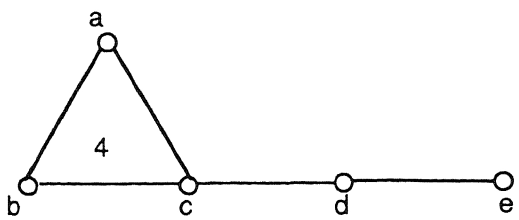

Figure 2.

Coxeter (1967) presented - among others - the following relation due to McMullen

$$[(abcd)^5, e] = 1.$$

Since the four reflections a, b, c, d generate an irreducible linear 4-dimensional group, the centre of the group they generate will be the centre of the whole group, whence also commute with the fifth generator e. The above relation emerges from the rule that in an n-dimensional linear group generated by a set Y of $n + 1$ reflections, the centres of the groups generated by a subset of Y of size n should commute with each member of Y. In the next section, we shall abide with this rule and the variation that central involutions obtained in this way should be identified.

5. Presentations of the quaternionic groups

We now come to the 7 presentations corresponding to the diagrams of Figure 1. They maintain the property that for each subset X of the nodes, a presentation of the corresponding reflection groups (generated by the reflections in X) can be obtained by disregarding all generators and relations containing a generator outside of X.

5.1 Q

The group $W(Q)$ is 3-dimensional, but cannot be generated by 3 reflections. The coset enumeration for the relations obtained from the 3-generator subgroups did not complete. The 3-generator subgroups $\langle a, b, d\rangle$ and $\langle b, c, d\rangle$ are both isomorphic to the well-known Coxeter group $W(B_3)$, and so have central involutions $(b\ a\ d)^3$ and $(b\ c\ d)^3$, respectively. But also $\langle a, c\ d\rangle \cong W(J_3(4))$ has a central involution: $(a\ c\ d)^7$, see §4.1. Now the centre identification rule comes into effect: we identify these central involutions, found in 3-generator subgroups. The following presentation results:

generators: $a, b, c, d;$

relations: $a^2 = b^2 = c^2 = d^2 = 1,$

$$(ab)^4 = (ac)^3 = (ad)^3 = (bc)^3 = (bd)^2 = (cd)^4 = 1,$$

$$(abcb)^3 = (acdc)^4 = 1,$$

$$(bad)^3 = (bcd)^3 = (acd)^7.$$

Coset enumeration with respect to the subgroup $\langle a, c, d \rangle \cong W(J_3(4))$ gives that the latter subgroup has index 36 in the presented group. Thus the presented group has the same order as its homomorphic image $W(Q)$ and so is isomorphic to it. We conclude that the above is a presentation for $W(Q)$. The centre of order 2 in the presented group is generated by *ababdabad*.

5.2 R

For R, observations similar to those for Q seem to apply. The subgroup generated by a, c, d is the complex reflection subgroup $W(J_3(5))$, so the involution $(a\ d\ c)^{15}$ is central and can again be identified with central involutions $(b\ a\ d)^3$ and $(b\ c\ d)^3$. Thus we are lead to consider the following presentation:

generators: $a, b, c, d;$

relations: $a^2 = b^2 = c^2 = d^2 = 1,$

$$(ab)^4 = (ac)^3 = (ad)^3 = (bc)^3 = (bd)^2 = (cd)^4 = 1,$$

$$(abcb)^3 = (acdc)^5 = 1,$$

$$(bad)^3 = (bcd)^3 = (adc)^{15}.$$

But a new phenomenon presents itself: The reflection group $W(R)$ is perfect, whereas the above presentation has relations of even length only, so that the subgroup generated by all products in a, b, c, d of even length is a subgroup of index 2. (Over a commutative field, of course, the reflections would have had determinant -1 and so never generate a perfect group.) The coset enumeration however does complete giving index 1120 over $\langle a, c, d \rangle$, twice the index the reflection subgroup $W(J_3(5))$ generated by the corresponding reflections has in $W(R) \cong 2 \cdot HJ$. Thus a single relation of odd length would suffice, or for that matter, a way to express a reflection as a product of commutators. The last line of the following presentation does that job (admittedly not in a very pretty way) providing a presentation for $W(R)$:

generators: $a, b, c, d;$

relations: $a^2 = b^2 = c^2 = d^2 = 1,$

$$(ab)^4 = (ac)^3 = (ad)^3 = (bc)^3 = (bd)^2 = (cd)^4 = 1,$$

$$(abcb)^3 = (acdc)^5 = 1,$$

$$acdcadcdacdacdacdcadcdacababcdacdabcdacdabcab = 1.$$

5.3 S_1

The group $W(S_1)$ is 4-dimensional. Thus, no additional rules can be derived from identification of central elements of subgroups.

Since the relations for the complex reflection subgroups did not lead to completion of the coset enumeration, an additional rule is called for. Observing that the diagram has a single 4-circuit, we have extended the labelling of triangles to one for all circuits. Here we label the full circuit by a 3 (not written in the diagram of Figure 1) to remind us of the additional relation $((aba)(dcd))^3 = 1$.

Thus we arrive at the following presentation:

generators: $a, b, c, d;$

relations: $a^2 = b^2 = c^2 = d^2 = 1,$

$$(ab)^3 = (ac)^3 = (ad)^2 = (bc)^3 = (bd)^3 = (cd)^3 = 1,$$

$$(abcb)^3 = (acdc)^3 = 1,$$

$$(badcda)^3 = 1.$$

Enumeration of cosets of $\langle b, c, d\rangle$ yields index 128. Since $\langle b, c, d\rangle \cong G(3, 3, 3) \cong 3^2{\cdot}Sym_3$, this yields the right order for $\langle a, b, c, d\rangle$ to coincide with $W(S_1)$. Consequently, the above is indeed a presentation of $W(S_1)$.

5.4 S_2

Proceeding analogously to the former case, we find that the following is a presentation for $W(S_2)$:

generators: $a, b, c, d;$

relations: $a^2 = b^2 = c^2 = d^2 = 1,$

$$(ab)^3 = (ac)^3 = (ad)^3 = (bc)^3 = (bd)^2 = (cd)^3 = 1,$$

$$(abcb)^3 = (acdc)^4 = 1,$$

$$(adbcbd)^3 = 1.$$

We observe that the element $abcd$ has order 24 and that its 12-th power generates the centre (a group of order 2). This fact plays a role in the presentation for $W(S_3)$ below.

5.5 S_3

The diagram of S_3 contains 5 nodes, while the group $W(S_3)$ is 4-dimensional. Thus the central element $(abcd)^{12}$ of $W(S_3)$ is central in the whole group. Adding this relation to the usual rules for the presentation, we obtain:

generators: a, b, c, d, e

relations: $a^2 = b^2 = c^2 = d^2 = e^2 = 1,$

$$(ab)^3 = (ac)^3 = (ad)^3 = (ae)^2 = (bc)^3 = (bd)^2 = (be)^3$$

$$= (cd)^3 = (de)^2 = (ce)^2 = 1,$$

$$(abcb)^3 = (acdc)^4 = 1,$$

$$(adbcbd)^3 = 1,$$

$$[e, (abcd)^{12}] = 1.$$

Enumeration of cosets with respect to the subgroup $\langle a, b, c, e, (adcd)^2\rangle$ showed that the group with this presentation has isomorphism type $2{\cdot}2{\cdot}((2^8 \cdot 2^6)\times 2){\cdot}PS\Omega^-(6, 2)$, while $W(S_3) \cong 2 \cdot 2^6 \cdot PS\Omega^-(6, 2)$. Since $W(S_3)$ is perfect, we should add a relation of odd length; we take the relation $(abcde)^5 = 1$. But then the resulting group still has a normal 2-subgroup of order 2^8 which does not occur in $W(S_3)$. This shows that the rules of thumb introduced so far do not suffice. To kill the latter normal 2-group, we had to add one more relation. The resulting presentation for $W(S_3)$ is:

generators: a, b, c, d, e

relations: $a^2 = b^2 = c^2 = d^2 = e^2 = 1,$

$$(ab)^3 = (ac)^3 = (ad)^3 = (ae)^2 = (bc)^3 = (bd)^2 = (be)^3 = (cd)^3 = (de)^2 = (ce)^2 = 1,$$

$$(abcb)^3 = (acdc)^4 = 1,$$

$$(adbcbd)^3 = 1,$$

$$[e, (abcd)^{12}] = 1,$$

$$dc(acd)^4cd = (be(abc)^2ebadc)^6,$$

$$(abcde)^5 = 1.$$

5.6 *T*

Exploiting the 4-circuit rule for the 4-dimensional group $W(T)$, the following presentation has been found:

generators: a, b, c, d

relations: $a^2 = b^2 = c^2 = d^2 = 1,$

$$(ab)^3 = (ac)^3 = (ad)^5 = (bc)^3 = (bd)^2 = (cd)^3 = 1,$$

$$(abcb)^3 = (acdc)^3 = 1,$$

$$(adbcbd)^3 = 1.$$

Coset enumeration with respect to $\langle a, c, d\rangle \cong G(5, 5, 3)$ gives index 17280 $= 2^7 3^3 5$. Since $G(5, 5, 3)$ has order $5^2 \cdot 3!$, the order of the presented group is $2^7 3^4 5^3$, which coincides with the order of $W(T)$. Therefore, the above is indeed a presentation of $W(T)$.

5.7 *U*

Finally we treat the 5-dimensional group $W(U)$. The 4-circuit rule suffices to find a satisfactory presentation for $W(U)$:

generators: a, b, c, d, e

relations: $a^2 = b^2 = c^2 = d^2 = e^2 = 1,$

$$(ab)^3 = (ac)^3 = (ad)^2 = (ae)^2 = (bc)^3 = 1,$$

$$(bd)^3 = (be)^2 = (cd)^3 = (de)^3 = (ce)^2 = 1,$$

$$(abcb)^3 = (bcdc)^3 = 1,$$

$$(cdabad)^3 = 1.$$

The fact that the presented group is isomorphic to $W(U)$ follows from the fact that the index 3960 has been found in enumerating the cosets of the subgroup $\langle a, b, c, d\rangle$ (isomorphic to $W(S_1)$) of the presented group.

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